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AN ANALYTIC MODEL FOR THE COMPRESSION OF PLASMA TOROIDS

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WEAPONS LABORATORY

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An expression for the energy required to compress a plasma toroid in the MARAUDER experiment has been derived. The analysis assumes that the plasma follows the ideal magneto-hydrodynamics approximation, that the magnetic forces are the only ones of interest, and that the toroid is in a confined equilibrium at the start and end of the compression. All dimensions of the toroid are forced to collapse linearly, and the magnetic energy in the toroid is found to be inversely proportional to the inner radius-- or any other dimension. This analysis gives an estimate to help limit the parameter space of a computational study of the compression process.

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PREFACE

This report describes analytic modeling of the MARAUDER experiment at the Weapons Laboratory. The research program was initiated by Dr. Kirk Hackett, who is currently working at the Air Force European Office of Aerospace Research and Development in London, England. The author wishes to acknowledge his help in resolving many issues related to the project. The author also wishes to thank Prof. Peter Turchi of Ohio State University. He spent the summer of 1990 at the Weapons Laboratory under the Inter-governmental Personnel Act and increased the Division's knowledge of the physics involved in the project through many informative discussions. Finally, Maj. Clifford Rhoades, USAFR, reviewed this report in detail, and his constructive criticism helped tremendously.

CONTENTS

1.0 INTRODUCTION

The RACE (Ring ACceleration Experiment) project at Lawrence Livermore National Laboratory has demonstrated that plasma toroids can be accelerated as armatures in a coaxial rail gun configuration (Ref. 1). The magnetic fields of these toroids are similar to those within spheromaks (Ref. 2) except these toroids remain off the axis of symmetry between a pair of coaxial conducting cylinders that serve as the electrodes for the rail gun. The MARAUDER (Magnetically Accelerated Rings to Achieve Ultra-high Density Energy and Radiation) project at the Weapons Laboratory High Energy Plasma Division (AWX) will attempt to carry the concept to the megajoule energy level. There are many possible applications for high energy toroids. They may be used to produce intense X-ray radiation by interrupting their motion with a metal plate or focusing cone. They may provide reliable armatures for plasma flow switches that can be used in pulsed-power devices for current shaping. There are also some exotic proposals for accelerated toroids including possibilities for fusion (Ref. 3).

The full MARAUDER experiment will be a three-stage process. The first stage is the formation of the torus and is performed with a magnetized plasma gun. This is similar to the spheromak formation that is described by Turner (Ref. 2). The second stage compresses the torus to a smaller diameter, increasing the plasma density and magnetic induction. The final stage is axial acceleration of the toroid. The three stages are represented schematically in Figure 1.

This report describes an analytic model of the compression stage. The model is presently being used to supplement time-dependent two-dimensional magneto-hydrodynamic (MHD) simulations that are performed on computers. The information from the two approaches will be used in the design of the compression cone and acceleration circuit. The analytic model gives the magnetic field and associated magnetic energy for a torus before and after compression. The increase in torus energy plus the field energy stored behind the torus is an estimate of what the acceleration circuit must provide. Section 2.0 describes the solutions for the magnetic field and integrated energy. Section 3.0 presents the method used for finding the eigenvalue for the MARAUDER geometry, and Section 4.0 derives a model from which comes the energy estimate.

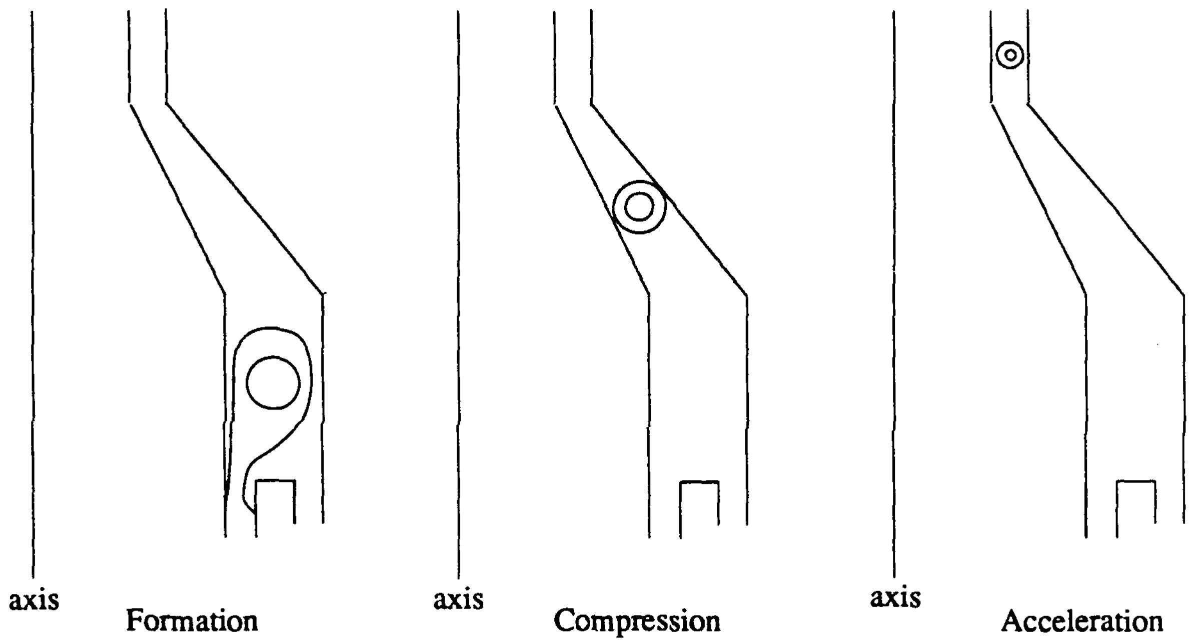


Figure 1. Schematic of the poloidal magnetic flux during the three stages of the MARAUDER experiment.

2.0 ANALYTIC SOLUTION

In low-β plasmas where hydrostatic pressure is small compared to magnetic field pressure, a reasonable assumption is that the only forces of interest are those from the magnetic fields. These plasmas are "force-free" when the plasma current density is nonexistent or everywhere parallel to the magnetic induction. The latter condition may be expressed as

$$\nabla \times \overline{B} = \lambda(\overline{x})\,\overline{B} \tag{1}$$

where $\overline{B}$ is the magnetic induction vector. The only restriction on this equilibrium configuration is that $\overline{B} \cdot \nabla\lambda(\overline{x}) = 0$ to enforce $\nabla \cdot \overline{B} = 0$. Woltjer (Ref. 3) has shown that the integral of the dot product of $\overline{B}$ and $\overline{\mathcal{A}}$, the magnetic vector potential, over the volume of a closed system is conserved when the system obeys the ideal MHD approximation. This integral is now known as the magnetic helicity, and Woltjer also proved that minimizing the magnetic energy of such a system keeping its helicity constant leads to Equation 1 with constant λ. A minimum energy configuration is stable, and the fields within spheromaks and the toroids of RACE and MARAUDER have such configurations. It is also useful to note that $\overline{\mathcal{A}}$ satisfies $\nabla \times \overline{\mathcal{A}} = \overline{\mathcal{A}}\lambda + \nabla\phi$, where the gauge determines ϕ, so the magnetic energy, $(1/2\mu_0)\int \overline{B} \cdot \overline{B}\, d^3x$, is $\lambda/2$ times the helicity (Ref. 4).

Finding the solution of Equation 1 with constant λ subject to appropriate boundary conditions is not new (Refs. 4 and 5, for example). The method is presented here for completeness. The energy density of the resulting fields is then integrated for the compression analysis. The conventions of Finn, Manheimer and Ott (Ref. 4) are used except cylindrical symmetry is assumed, so the scalar variations with respect to the azimuthal coordinate are zero. This assumption implies but does not prove that a symmetric configuration has a smaller λ, hence lower energy, than any other configuration.

The solution of Equation 1 with constant λ may be obtained from the solution of the scalar Helmholtz equation,

$$\nabla^2\psi + \lambda^2\psi = 0 \tag{2}$$

The magnetic induction vector is then determined by

$$\overline{B} = \overline{z} \times \nabla\psi + (1/\lambda)\nabla \times (\overline{z} \times \nabla\psi) \tag{3}$$

where $\overline{z}$ is the axial unit vector in cylindrical coordinates. Morse and Feshbach (Ref. 6) cover this technique in detail for generalized vector fields. The domain of this model is an annular volume that forms a rectangle when sliced by the r-z plane--see Figure 2. The rectangle has two

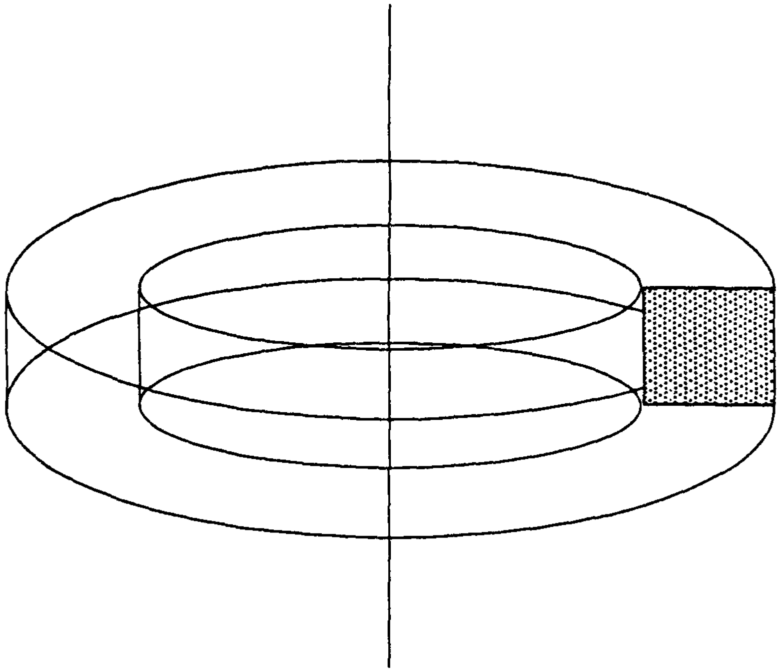


Figure 2. The rectangular annulus that forms the domain of the analytic model and its intersection with the r-z plane.

sides parallel to $\overline{z}$. The boundary conditions that make $\overline{B}$ tangent to the bounding surfaces are

$$\frac{\partial^2 \psi}{\partial r \partial z} = 0 \tag{4}$$

at $r = r_1$ and $r = r_2$, and

$$\frac{\partial^2 \psi}{\partial z^2} + \lambda^2 \psi = 0 \tag{5}$$

at z=0 and z=L, where r is the radial coordinate, r_1 and r_2 are the inner and outer radii of the

formation section (see Figure 1), and L is the axial length. Equation 2 may be solved by separation of variables, setting $\psi(r,z) = R(r)Z(z)$, and the differential equation becomes

$$\frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \lambda^2 = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

The two sides of the equation must be equal to a constant α^2. The resulting ordinary differential equation for Z has a solution that is the sum of a sine function and a cosine function,

$$Z = C_s \sin(\alpha z) + C_c \cos(\alpha z)$$

The boundary conditions then require

$$C_s(\lambda^2 - \alpha^2)\sin(\alpha z) + C_c(\lambda^2 - \alpha^2)\cos(\alpha z) = 0$$

at $z = 0$, $z = L$. To avoid restricting the rest of the solution, the condition at $z = 0$ requires $C_c = 0$. A series of eigenvalues, $\alpha_n = n\pi/L$ with $n = 1, 2, \ldots$, for a series of sine terms, will then satisfy the boundary conditions.

The solution of the differential equation for R is a sum of linearly independent Bessel functions of order zero. Before applying the boundary conditions at the radial boundaries, the solution for ψ is

$$\psi(r,z) = \sum_{n=1}^{\infty} C_n[J_0(r\sqrt{\lambda^2 - \alpha_n^2}) + A_n Y_0(r\sqrt{\lambda^2 - \alpha_n^2})]\sin(\alpha_n z) \qquad (6)$$

If $r_1 = 0$ to create a spheromak, then the boundary condition at r_1 forces $A_n = 0$ for all n. This is not appropriate for coaxial configurations. Substituting Equation 6 into Equation 4 and evaluating at the two radial boundaries gives two series that are each zero. The differential equation for the minimum energy configuration has a constant λ, and this can be satisfied when C_n is nonzero for only one n. Because the energy is the helicity, which is fixed, multiplied by $\lambda/2$, the configuration with the smallest λ has the lowest energy. The arguments of the Bessel functions must be real for the cylindrically symmetric case, as the modified Bessel functions that would arise from imaginary arguments are monotonic and would not satisfy Equation 4 at both radii. Therefore, $\lambda > \alpha_n$. Solving the n=1 terms of the two series with $\alpha = \pi/L$,

$$J_1(r_1\sqrt{\lambda^2 - \alpha^2}) + AY_1(r_1\sqrt{\lambda^2 - \alpha^2}) = 0 \tag{7a}$$

and

$$J_1(r_2\sqrt{\lambda^2 - \alpha^2}) + AY_1(r_2\sqrt{\lambda^2 - \alpha^2}) = 0 \tag{7b}$$

for the smallest radial factor, $\sqrt{\lambda^2 - \alpha^2}$, and corresponding A gives the solution with the minimum energy. Determining the radial factor and A is the subject of Section 3.0.

Substituting Equation 6 into Equation 3 gives the components of $\overline{B}$. The constant is changed for convenience to $K = C_1\sqrt{\lambda^2 - \alpha^2}$, and the components are

$$B_r = (K\alpha/\lambda)\cos(\alpha z)[J_1(r\sqrt{\lambda^2 - \alpha^2}) + AY_1(r\sqrt{\lambda^2 - \alpha^2})] \tag{8a}$$

$$B_\theta = -K\sin(\alpha z)[J_1(r\sqrt{\lambda^2 - \alpha^2}) + AY_1(r\sqrt{\lambda^2 - \alpha^2})] \tag{8b}$$

$$B_z = -(K\alpha\sqrt{\lambda^2 - \alpha^2}/\lambda)\sin(\alpha z)[J_0(r\sqrt{\lambda^2 - \alpha^2}) + AY_0(r\sqrt{\lambda^2 - \alpha^2})] \tag{8c}$$

The energy of this configuration is the volume integral of the magnetic energy density, $B^2/2\mu_0$. This yields

$$E_B = \frac{K^2\pi^2}{2\mu_0}\left\{\frac{\alpha^2 + \lambda^2}{\alpha\lambda^2(\lambda^2 - \alpha^2)}[(x^2/2)F^2(x) - xF(x)G(x) + (x^2/2)G^2(x)] + (1/\alpha\lambda^2)[(x^2/2)F^2(x) + (x^2/2)G^2(x)]\right\}\Bigg|_{x = r_1\sqrt{\lambda^2 - \alpha^2}}^{x = r_2\sqrt{\lambda^2 - \alpha^2}} \tag{9}$$

where

$$F(x) = J_0(x) + AY_0(x)$$

and

$$G(x) = J_1(x) + AY_1(x)$$

When A and λ satisfy the boundary conditions, G(x) is zero at the limits of integration, and the energy in the toroidal field is half of the total. The flux of toroidal field through the r-z plane is also necessary for the compression analysis. The area integral of Equation 8b gives

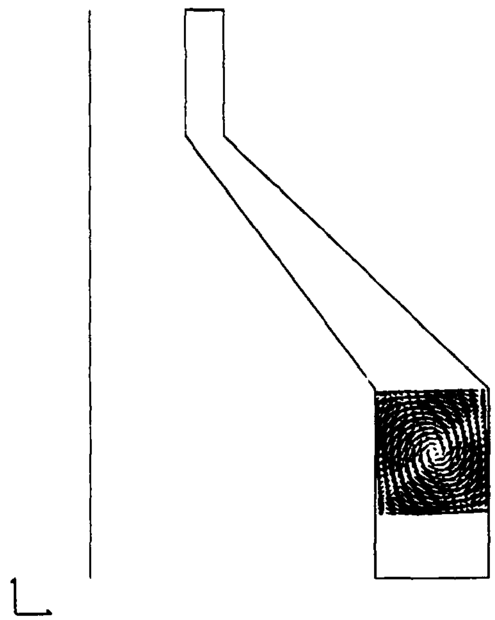
$$\Phi = K\frac{2}{\alpha\sqrt{\lambda^2 - \alpha^2}}F(x)\Bigg|_{x = r_1\sqrt{\lambda^2 - \alpha^2}}^{x = r_2\sqrt{\lambda^2 - \alpha^2}} \tag{10}$$

3.0 RADIAL BOUNDARY CONDITIONS

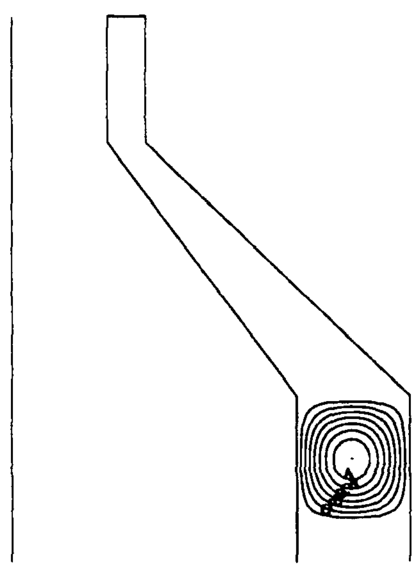
The two unknowns in Equations 7a and 7b are determined with a numerical root-solver using SLATEC library routines for the Bessel functions. First, a cost function is defined as the sum of the squares of the residuals from Equations 7a and 7b when guesses are substituted for A and λ. When the boundary conditions are satisfied, the cost function is a local minimum; in fact it will be zero. Some knowledge of the Bessel functions and the desired results helps. The Bessel functions are smooth, and the smallest radial factor is desired.

The program BESROOT (see the Appendix for a listing) provides a semi-automated approach to finding the roots in the two-dimensional space of A and λ. The user selects a set λ's, and the program finds the corresponding set of minimum costs between A = -10 and A = +10 using a Newton's method in the A dimension. A larger range of A is possible, but this would imply that the Bessel function of the second kind by itself would almost fit the geometry. Also, the search in λ could have been automated, but this would add much more complexity, and the manual search is not time consuming for one geometry.

The MARAUDER formation stage dictates $r_1 = 0.4477$ m and $r_2 = 0.6255$ m, the inner and outer radii, respectively, of the chamber. The smallest radial factor, $\sqrt{\lambda^2 - \alpha^2}$, that satisfies the boundary conditions is 17.744 m^{-1}, and the corresponding A is 1.3191. The axial length, L, of the toroid is somewhat arbitrary. The axial confinement of the plasma is a dynamic process in the experiment and is beyond the scope of this analysis. Good confinement would produce a toroid whose L is about the same as the radial gap between the conducting walls. Therefore, for one example, L is set to 0.2 m, and λ becomes 23.698 m^{-1}. Figure 3 shows the resulting field distribution serving a second role as initial conditions for a time-dependent computer simulation. This simulation has K = 1, which gives 4.388 kJ of magnetic field energy.



(a) Poloidal magnetic field vectors.



(b) Toroidal magnetic field contours.

Figure 3. The magnetic fields of the analytic model as initial conditions for a computer simulation.

4.0 COMPRESSION ANALYSIS

Several assumptions and approximations are necessary to gain useful information about the compression stage from the expressions that have been derived. The analysis provides the energy of the compressed state, and the assumptions deal with what the compressed state is without solving time-dependent equations. The preceding discussion has dealt with "force-free" magnetic fields within a plasma-filled rectangular annulus and has not addressed what exists outside of the annulus. If it were surrounded by a perfectly conducting surface, eddy currents would run along the surface. Material strength of the surface would be required to oppose the pressure of the poloidal field, which does not penetrate the surface. If there were no conducting surface, the configuration could remain in equilibrium only if vacuum fields matched the poloidal fields at the surface of the annulus. The experimental toroids of MARAUDER represent neither extreme, but they have elements of each. The radial boundaries of the chamber are only 18 cm apart in the formation section, and the total chamber length can be several meters. The concept is to form a confined plasma ring, and accelerate it through the chamber. The goal is not to fill the entire chamber with plasma. Vacuum fields and inertia help the axial confinement. The whole process is dynamic, and equilibrium is never established.

The compression process is approximated by forcing the torus to remain in a rectangular annulus whose dimensions collapse linearly towards a point on the axis. Different acceleration tube geometries may be explored by changing the final size of the theoretical annulus. The inner radius, r_i, of the annulus is a convenient parameter, and the outer radius, r_o, and axial length, l, are proportional to it:

$$r_o(r_i) = \chi r_i \text{, and } l(r_i) = \eta r_i$$

The constant η is determined by the formation chamber, and χ comes from the arbitrary choice for L. For the configuration to remain unchanged with various r_i, the arguments of the trigonometric functions and Bessel functions in Equations 8a through 8c must remain unchanged. Therefore,

$$\alpha(r_i) = \pi/\eta r_i \text{, and } \lambda(r_i) = \Lambda/r_i$$

where $\Lambda = r_1 \cdot \lambda(r_1)$. It is also convenient to evaluate the definite integral in the energy expression that satisfies the boundary conditions, because it will also remain unchanged for various r_i:

$$\Omega \equiv \frac{x^2}{2}[F(x)]^2 \Big|_{x = r_1\sqrt{\lambda^2 - \alpha^2}}^{x = r_2\sqrt{\lambda^2 - \alpha^2}}$$

Its value is 2.763 for MARAUDER.

If the plasma obeys ideal MHD, where the plasma acts as a perfectly conducting fluid, then the magnetic flux through a surface element of the plasma remains constant. Thus, the flux expression, Equation 10, implies $K(r_i) = K(r_1)(r_1/r_i)^2$. The magnetic energy in the torus is now a function of r_i only,

$$E_B = \frac{[K(r_1)]^2 \pi\eta\Omega r_1^4}{\mu_0(\Lambda^2 - \pi^2/\eta^2)r_i} \tag{11}$$

The magnetic energy of the idealized torus is inversely proportional to radius. A compression cone that has been proposed for MARAUDER reduces the radius by a factor of three, so the torus illustrated in Figure 3 would emerge from the cone with approximately 13 kJ of energy.

The acceleration circuit provides azimuthal magnetic field behind the torus to compress and accelerate it. The current in the acceleration circuit peaks when the torus reaches the end of the compression cone. The circuit is crowbarred at this point to efficiently accelerate the torus as the stored field expands adiabatically. An estimate of the energy the acceleration circuit must provide, E_c, for a given torus is the work done on the torus during compression plus the field energy stored behind the torus,

$$E_c = \Delta E_B + (1/2)(L_{in} + L_{ex})I^2 \tag{12}$$

The internal energy, kinetic energy and radiative losses are relatively small prior to acceleration. The inductance per unit length between concentric cones is the same as that between coaxial cylinders with the same ratio of radii, χ. Therefore the inductance internal to the chamber is

$$L_{in} = (\mu_0 h/2\pi)\ln(\chi)$$

where h is the axial distance between the bottom of the formation section and the end of the

compression cone. The inductance external to the chamber, L_{ex}, is determined experimentally by shorted discharges.

One final approximation is necessary to relate the peak current, I, to the compressed torus. If the torus at the end of the cone is in equilibrium with the field from the circuit, then the force from one balances that of the other. These forces do not have the same distribution, and the circuit field will distort the upstream side of the torus. However, if an imaginary conducting washer separates the torus from the circuit field, then the area integrals of magnetic pressure can be matched to provide an expression for I. The toroid has only radial field at the axial boundaries of the rectangular annulus, and using Equation 8a its force on the washer is

$$[K(r_1)]^2 \frac{\pi^3 \Omega r_1^4}{\mu_0 \eta^2 \Lambda^2 (\Lambda^2 - \pi^2/\eta^2) r_i^2}$$

and the force from the circuit is $(\mu_0 I^2/4\pi)\ln(\chi)$, so the current that balances a torus is

$$I = K(r_1) \frac{2\pi^2 r_1^2}{\eta \Lambda \mu_0 r_i} \left(\frac{\Omega}{\ln(\chi)(\Lambda^2 - \pi^2/\eta^2)} \right)^{1/2}$$

The complete expression for the required circuit energy is

$$E_c = \frac{[K(r_1)]^2 \pi \eta \Omega r_1^3}{\mu_0 (\Lambda^2 - \pi^2/\eta^2)} \left(\frac{r_1}{r_i} - 1 \right)$$
$$+ \frac{1}{2} \left(\frac{\mu_0 h}{2\pi} \ln(\chi) + L_{ex} \right) \frac{[K(r_1)]^2 4\pi^4 \Omega r_1^2}{\mu_0 \eta^2 \Lambda^2 (\Lambda^2 - \pi^2/\eta^2) \ln(\chi)} \left(\frac{r_1}{r_i} \right)^2 \qquad (13)$$

This is proportional to the square of the initial field, through the $K(r_1)$ factor, and it is a quadratic of the compression ratio, r_1/r_i, where the highest order term is the stored field energy behind the torus. Finally the length of the cone is a linear multiplier in the internal inductance term. With $h = 0.7$ m, and $L_{ex} = 50$ nH--one possible combination for MARAUDER, the peak circuit current for the toroid in Figure 3 is 1.610 MA, and the circuit must provide a total of 134 kJ.

5.0 DISCUSSION

This analysis for the compression of a torus is not a detailed model. The issue of axial confinement is neglected as the imaginary conducting surface in the model always encloses the plasma and magnetic fields. This is a gross approximation when one considers that the same configuration placed in the actual chamber (without surfaces on the axial boundaries) does not satisfy Ampere's Law. Therefore, the real torus must have vacuum fields surrounding it. However, the energy expression from the model does provide useful information. It gives a rough idea of what circuit parameters are legitimate for a given torus and cone geometry. The analysis is not complete, but it does help limit the parameter space of the computational study, saving time and computer resources.

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APPENDIX

```
      program besroot

c-----this program helps find the solution to the equation
c     J1(x) + a*Y1(x) = 0, where x = r*sqrt(mu**2-(pi/L)**2),
c     at two values of r.

      real sinev, besev, totev, evinc, r1, r2, a, amx, amn
      integer ita, ite

      common /radii/ r1, r2

      open( unit = 1, status = 'new', file = 'besout')
      call link('unit2=tty//')

c-----for L = 20 cm
      sinev = 15.708
c-----formation region
      r1 = 0.4477
      r2 = 0.6255
      amaxi = 10.
      amini = -10.
      ita = 40

  100 write(*,*) 'How many iterations of the eigenvalue, '
      write(*,*) 'what starting point, and increment?'
      read(2,*) ite, totev, evinc

      do 200 i1 = 1,ite
       amx = amaxi
       amn = amini
```

```
      besev = sqrt( totev**2 - sinev**2 )
      dcostmax = dcost(besev, amx)
      dcostmin = dcost(besev, amn)
      if ( (dcostmax * dcostmin) .lt. 0. ) then
c---------the pair stradle 0.
        do 300 i2 = 1,ita
          an = (amx + amn) / 2.
          dcostn = dcost(besev, an)
          if ( (dcostmax * dcostn) .lt. 0. ) then
            amn = an
            dcostmin = dcost(besev, amn)
          else
            amx = an
            dcostmax = dcost(besev, amx)
          endif
 300    continue
        write(*,*) totev, amn, valu(besev,amn,r1), valu(besev,amn,r2)
        write(1,*) totev, amn, valu(besev,amn,r1), valu(besev,amn,r2)
      else
        write(*,*) totev, 'No root between', amn, amx
        write(1,*) totev, 'No root between', amn, amx
      endif
      totev = totev + evinc
 200  continue
      write(*,*) 'Type 1 for another round.'
      read(2,*) irep
      if (irep .eq. 1) goto 100
      close(unit = 1)

      stop
      end
```

```
      function dcost(ev, var)

c-----derivative of the cost function with respect to var
c     where the cost is the sum of the squares of the residual
c     of the function in the first comment

      common /radii/ r1, r2

      besj11 = besj1( ev*r1 )
      besj12 = besj1( ev*r2 )
      besy11 = besy1( ev*r1 )
      besy12 = besy1( ev*r2 )
      dcost =  2. * ( besj11 + var * besy11 ) * besy11 +
     %         2. * ( besj12 + var * besy12 ) * besy12

      return
      end


      function valu(ev, var, r)

c-----evaluate the function in the first comment

      arg = ev * r
      valu = besj1(arg) + var * besy1(arg)

      return
      end
```